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On minimal cutsets in P_5-free minimal imperfect graphs

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Abstract

Let G be a minimal imperfect P_5-free graph (i.e. a minimal imperfect graph not containing a path on 5 vertices as induced subgraph) and let S be a minimal cutset of G. In this paper we study several properties of such cutsets, in particular we prove that the subgraph induced by S is connected, contains a P_4, cannot induce a bipartite subgraph of $G, \ldots$. As a by-product we also give a structural characterization of such graphs. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

A graph is *perfect* if the vertices of any induced subgraph H can be colored, in such a way that no two adjacent vertices receive the same color, with a number of colors (denoted by $\chi(H)$) not exceeding the cardinality $\omega(H)$ of a maximum clique of H. For a graph G we denote by $\alpha(G)$ the cardinality of a maximum stable set of $V(G)$.

A graph is *minimal imperfect* if all its proper induced subgraphs are perfect but it is not. In particular, $\omega(G) + 1 = \chi(G)$ for G minimal imperfect. All the notions not defined here may be found in [2].

It is an easy task to check that an odd chordless cycle of length at least five (usually called a *hole*), as well as its complement (usually called an *anti-hole*) are minimal imperfect graphs.

The remark above and some early results concerning perfect graphs determined Berge [1] to formulate the two following conjectures (known as the Strong and the Weak Perfect Graph Conjecture)

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- (SPGC) A graph is perfect if and only if it does not contain an odd hole or an odd anti-hole as an induced subgraph.
- (WPGC) A graph is perfect if and only if its complement is perfect.

While the SPGC is still unsettled, the WPGC is an easy consequence of the following theorem of Lovász [8]:

Theorem 1 (The Perfect Graph Theorem). *A graph $G=(V,E)$ is perfect if and only if for every induced subgraph H of G the following inequality holds*:

$$\alpha(H)\cdot\omega(H)\geqslant|H|.$$

This theorem was the first step towards a new approach of minimal imperfect graphs. We can deduce from this theorem, that in a minimal imperfect graph G,

(1) $n=\alpha\cdot\omega+1$.
(2) For every vertex $v\in V(G)$, $G-v$ has a unique partition into $\alpha\omega$-cliques (i.e. a clique of size ω) and a unique partition into ω α-stable sets.
(3) Each vertex of G is in exactly α α-stable sets and in exactly ω ω-cliques.

Bland et al. [3] defined a graph to be *partitionable* if there exist two numbers $\alpha,\omega\geqslant 2$ such that (1) and (2) hold. Further refinements along this line are due to Padberg [13]. We only need the following property:

Proposition 2. *In a minimal imperfect graph G, given two vertices u and v, there exists an ω-clique containing u and not containing v.*

Proof. Use property (3). □

Two of the most useful graphical properties of minimal imperfect graphs were found by Meyniel and Chvátal. To describe their results we need introduce a few definitions.

Two nonadjacent vertices x,y form an *even pair* if all chordless paths joining x to y have an even number of edges. A set C of vertices of a connected graph G is called a *star-cutset* if G–C is not connected and there is a vertex x in C adjacent to all other vertices of C; x is called the *center* of C.

Lemma 3 (Meyniel [12]). *No minimal imperfect graph contains an even pair.*

Lemma 4 (Chvátal [4]). *No minimal imperfect graph contains a star-cutset.*

Now, we denote by $N_G(x)$ the set of vertices of G adjacent to x; when there can be no confusion we shall write $N(x)=N_G(x)$.

Lemma 5. *No minimal imperfect graph contains two nonadjacent vertices u and v such that*

$$N(v)\subseteq N(u)\quad or\quad N(u)\subseteq N(v)$$

(*one says they have comparable neighbourhoods*).

Proof. Assume, for example, that $N(v) \subseteq N(u)$ then $S = \{u\} \cup N(v)$ is a star-cutset except if $V = \{u, v\} \cup N(u)$ and $N(u) = N(v)$ (because $|V \setminus S| = 1$) but in this case $\{u, v\}$ form an even-pair, which is impossible. □

We now introduce another useful tool.

Definition 6. Let $G = (V, E)$ be a minimal imperfect graph and let u, v be two nonadjacent vertices of G. We denote by $G + uv$ the graph $(V, E \cup \{uv\})$ and one says that u, v is a *co-critical* pair if: $\omega(G + uv) = \omega(G) + 1$. Analogously, an edge $xy \in E$ is said to be *critical* if removing this edge from E increases the stability number of G.

Giles et al. [7] (see also Markossian and Karapetian [11]) prove this interesting result:

Theorem 7 (Giles et al. [7], Markossian and Karapetian [11]). *No minimal imperfect graph* (*different from an odd hole or an odd anti-hole*) *contains a cycle of critical edges.*

We are interested in the following consequence of their result :

Lemma 8. *No minimal imperfect graph* (*different from an odd hole or an odd anti-hole*) *contains three vertices inducing a stable set of co-critical pairs.*

There are many conjectures concerning minimal imperfect graphs, in particular Chvátal [5] proposes:

Conjecture 9. Every minimal imperfect graph G containing no odd hole and no odd antihole has the following properties:

(1) For each cutset C of G, the subgraph $[C]_G$ of G induced by C is connected.
(2) For each cutset C of G, the subgraph $[C]_G$ contains a P_4.

The main purpose of this paper is to prove this conjecture for P_5-free graphs. As a by-product we obtain a structural characterization of minimal imperfect and P_5-free graphs.

2. On minimal cutsets

We shall study properties of minimal cutsets in minimal imperfect P_5-free graphs. We call the *complete join* of two (disjoint) graphs $A = (V(A), E(A))$ and $B = (V(B), E(B))$ the graph with the vertex set $V(A) \cup V(B)$ and the edge set $E(A) \cup E(B) \cup \{ab | a \in V(A), b \in V(B)\}$. A graph is called *Berge* if it contains no odd hole and no odd antihole. We will prove the following results in part 4 of the paper:

Lemma 10. *Let G be a minimal imperfect and (P_5, C_5)-free graph, and let S be a minimal cutset of G. If $[S]_G$ is the complete join of two graphs A and B (with $V(A) \neq \emptyset$* but *$V(B)$ may be empty) then A is isomorphic to a connected subgraph of G.*

Remark 11. When we perform the complete join of a graph A and a graph B such that $V(A) \neq \emptyset$ and $V(B) = \emptyset$, the result is the graph A. So, a minimal cutset S of G induces a connected subgraph of G, else we can choose $V(A) = S$ and $V(B) = \emptyset$ which contradicts Lemma 10.

So, we have

Theorem 12. *If G is a minimal imperfect and (P_5, C_5)-free graph then for each minimal cutset S of G, the subgraph of G induced by S is connected.*

Theorem 13. *If G is a minimal imperfect and (P_5, C_5)-free graph then for each minimal cutset S of G, the subgraph of G induced by S contains a P_4.*

We know that, in a minimal imperfect graph, the neighbourhood of any vertex is a minimal cutset. So, we have the following corollaries:

Corollary 14. *If G is a minimal imperfect and (P_5, C_5)-free graph, then for each vertex v, the subgraph induced by $N(v)$ is connected.*

Remark 15. One can show that this property is also true if the graph G is partitionable and (P_5, C_5)-free.

Corollary 16. *If G is a minimal imperfect and (P_5, C_5)-free graph, it cannot have a vertex v such that the subgraph of G induced by $N(v)$ is P_4-free.*

Corollary 17. *If G is a (P_5, C_5)-free graph such that every its induced subgraph H has a vertex v such that the subgraph of H induced by $N_H(v)$ is P_4-free then G is perfect.*

Remark 18. This class of perfect graphs is a subclass of Slightly Triangulated Graphs the perfection of which was shown in [10].

We can also derive from Lemma 10 the following characterization of minimal imperfect and (P_5, C_5)-free graphs. Let $G = (V, E)$ be such a graph, let v be a vertex of G and let $M(v) = M_G(v) = N_{\bar{G}}(v) = \{u \in V | u \neq v, uv \notin E\}$. For $w \in N(v)$ we put $\mathscr{M}(w) = M(v) \cap N_G(w)$. For a subset Y of V we write $N_Y(u) = N(u) \cap Y$.

Theorem 19. *For every vertex $v \in V(G)$, one has a partition of $V(G)$ in*

$$\{v\}, W, Y, \text{ and } M$$

such that $N(v)$ is partitioned in W and $[Y]_G$ (with $Y \neq \emptyset$, $[Y]_G$ is connected and $W = \{w\} \cup N_{N(v)}(w)$, *where w is a vertex such that $M = \mathscr{M}(w)$).*

Now, let $\mathscr{B}$ be the family of bipartite graphs and let $\mathscr{B}^*$ be the family defined by

- $\mathscr{B} \subseteq \mathscr{B}^*$.
- $\forall G_1, G_2 \in \mathscr{B}^*$, the complete join and the disjoint union of G_1 and G_2 are in $\mathscr{B}^*$.

Gallai [6] proved that, for each vertex v of a minimal imperfect graph G, the neighbourhood of v induces a connected subgraph of the complement of G; hence if $[N(v)]_G \in \mathscr{B}^*$, then we have $[N(v)]_G$ is disconnected or $N(v)$ induces a bipartite graph (the latter case is impossible if G is Berge because this implies that $\omega(G) = 3$ and Tucker [15] has shown that SPGC is true for K_4-free graphs).

Then, if Lubiw's conjecture [9] (in G minimal imperfect and Berge, $\forall v \in V(G)$, $[N(v)]_G$ is connected) is true, for each vertex v of G we have $[N(v)]_G \notin \mathscr{B}^*$. So, we propose this weaker conjecture:

Conjecture 20. If G is a minimal imperfect Berge graph then for every vertex v of G we have $[N(v)]_G \notin \mathscr{B}^*$.

This conjecture implies that $[N(v)]_G$ contains a P_4. In the case of minimal imperfect and P_5-free graphs, we prove this conjecture for every minimal cutset. For doing this, our key lemma is the following:

Lemma 21. *Let G be a P_5-free minimal imperfect Berge graph and let S be a minimal cutset of G. If $[S]_G$ is the complete join of two graphs A and B ($V(A) \neq \emptyset$, but $V(B)$ may be empty), then A is not isomorphic to a bipartite subgraph of $[S]_G$.*

3. Preliminary results

Let $G = (V, E)$ be a minimal imperfect and (P_5, C_5)-free graph. Let $v \in V$ be a vertex of G.

Lemma 22. *Let G be a (P_5, C_5)-free graph. Let $v \in V$ and $x, y \in N(v)$ such that $xy \notin E$ then $\mathscr{M}(x) \subseteq \mathscr{M}(y)$ or $\mathscr{M}(y) \subseteq \mathscr{M}(x)$ (if $\mathscr{M}(x)$ and $\mathscr{M}(y)$ are nonempty).*

Proof. Suppose that there exist $x, y \in N(v)$ contradicting the hypothesis. So, there exist $a \in \mathscr{M}(x) \setminus \mathscr{M}(y)$ and $b \in \mathscr{M}(y) \setminus \mathscr{M}(x)$, but $\{a, x, v, y, b\}$ induces a P_5 or a C_5, a contradiction. $\square$

Now, let S be a minimal cutset of G, and let $G_1, G_2, \dots, G_q$ $(q \geqslant 2)$ be the connected components (meant as maximal subsets of vertices inducing connected subgraphs) of $G \setminus S$. We say that a vertex $x \in V$ is *incomplete* for a set A if we can find a vertex $y \in A$ such that $xy \notin E$ otherwise vertex x is said to be *complete* for A.

Lemma 23. *For every vertex $v \in S$, v is incomplete for, at most, one component G_i $(1 \leqslant i \leqslant q)$.*

Proof. Suppose that $v \in S$ is incomplete for two components G_i and G_j $(i \neq j)$. Each component is connected and then, one can find a vertex v_i in $N_{G_i}(v) \neq \emptyset$ and a vertex w_i in $V(G_i) \setminus N_{G_i}(v) \neq \emptyset$ such that $v_i w_i \in E$ (we can do this for G_j) but then the subset $\{w_i, v_i, v, v_j, w_j\}$ induces a P_5. $\square$

Remark 24. Let v be a vertex of G_i $(1 \leqslant i \leqslant q)$ then for every vertex x in S we have $\mathscr{M}(x) \neq \emptyset$ $(G_j \subseteq \mathscr{M}(x)$ for $j \neq i)$. So from now on, we will not verify if $\mathscr{M}(x)$ and $\mathscr{M}(y)$ are nonempty when applying Lemma 22.

4. The proofs

Proof of Lemma 10. Let G be a minimal imperfect and (P_5, C_5)-free graph, let S be a minimal cutset of G and let $G_1, G_2, \dots, G_q$ $(q \geqslant 2)$ be the connected components of $G \setminus S$. Suppose that $[S]_G$ is the complete join of two graphs A and B $(V(A) \neq \emptyset$, and $V(B)$ may be empty) such that A is isomorphic to a disconnected subgraph (we denote it also by A) of $[S]_G$. Let $A_1, A_2, \dots, A_p$ $(p \geqslant 2)$ be the connected components of A.

Case 1: $q \geqslant 3$. Let $x \in V(A)$ be a vertex such that $N_G(x) - S$ is maximal. We can suppose, without loss of generality, that $x \in A_1$. Then for every vertex y in A_2, the neighbourhood of y in $\bigcup_{1 \leqslant i \leqslant q} G_i$ is included in the neighbourhood of x in $\bigcup_{1 \leqslant i \leqslant q} G_i$. Indeed, since $q \geqslant 3$ there exists one index k $(1 \leqslant k \leqslant q)$ such that x and y are complete for G_k (Lemma 23). Let $v \in G_k$, we have $\bigcup_{i \neq k} G_i \subseteq M(v)$ so if we apply Lemma 22 with this vertex v, we have $\mathscr{M}(x) \subseteq \mathscr{M}(y)$ or $\mathscr{M}(y) \subseteq \mathscr{M}(x)$ and, particularly,

$$N_{\cup_{i \neq k} G_i}(x) \subseteq N_{\cup_{i \neq k} G_i}(y) \quad \text{or} \quad N_{\cup_{i \neq k} G_i}(y) \subseteq N_{\cup_{i \neq k} G_i}(x),$$

that is $N_{\cup_i G_i}(x) \subseteq N_{\cup_i G_i}(y)$ or $N_{\cup_i G_i}(y) \subseteq N_{\cup_i G_i}(x)$ since x and y are complete for G_k. But $N_{\cup_i G_i}(x)$ is maximal, therefore $N_{\cup_i G_i}(y) \subseteq N_{\cup_i G_i}(x)$.

If x is incomplete for a component G_i then $\{x\} \cup N_G(x)$ forms a star disconnecting subgraphs induced by subsets A_2 and $G_i \setminus N_{G_i}(x)$. So, we can suppose that x is complete for $G_1, G_2, \dots, G_q$ but in this case, if $V(A) \neq \{x\} \cup A_2$, the subset $\{x\} \cup G_1 \cup G_2 \cup \cdots \cup G_q \cup V(B)$ forms a star cutset. Then we only need to study the following case:

Claim 25. *If $[S]_G$ is the complete join of two graphs A and B then we cannot have*

- $V(A) = \{x\} \cup Y$ (*where A is disconnected and the subgraph of A induced by Y is connected*), *and*
- *x complete for $G_1 \cup \cdots \cup G_q$ $(q \geqslant 2)$.*

The subset Y can be partitioned into $U, Y_1, \ldots, Y_q$ where

- U: vertices in Y which are complete for $G_1, \ldots, G_q$;
- Y_i $(1 \leqslant i \leqslant q)$: vertices in Y which are incomplete for G_i.

We first show that $U = \emptyset$. We assume that $y \in U \subseteq Y$ is complete for $G_1, \ldots, G_q$ $(q \geqslant 2)$ and show that this leads to a contradiction:

(a) If $Y - \{y\} \neq \emptyset$ then $\{y\} \cup G_1 \cup \cdots \cup G_q \cup V(B)$ forms a star-cutset.
(b) If $Y - \{y\} = \emptyset$ then x and y have the same neighbourhood (which contradicts Lemma 5).

Now, assume there exist two indices i and j $(i \neq j)$ such that $Y_i \neq \emptyset$ and $Y_j \neq \emptyset$. Let $u \in Y_i$, $v \in Y_j$, $t \in G_j \setminus N(v) \neq \emptyset$ and $r \in G_i \setminus N(u) \neq \emptyset$ then the subset $\{u, t, x, r, v\}$ induces a P_5 or a C_5. So, we can suppose, without loss of generality, that $Y_1 \neq \emptyset$ and that for each j, $2 \leqslant j \leqslant q$, $Y_j = \emptyset$ (i.e. $Y = Y_1$). Moreover, no vertex y in Y is *universal* for Y (that is adjacent to all vertices in Y), otherwise

(a) If $\omega([G_2]_G) = \omega - 1$, then any ω-clique containing $v \in G_2$ also contain x or y which implies $Y = \{y\}$ (because the vertices in Y are complete for G_2). But, then x and y have comparable neighbourhood (Lemma 5).
(b) Let $\omega([G_2]_G) \leqslant \omega - 2$
  - If $|Y| \geqslant 2$ then any ω-clique containing $v \in G_2$, contains y which is impossible (Proposition 2).
  - If $Y = \{y\}$ then x and y have comparable neighbourhood (Lemma 5).

Lastly, pick up a vertex y in Y such that $N(y) - S$ is maximal. We know that $\forall y' \in Y$ such that $yy' \notin E$, $N_{G_1}(y') \subseteq N_{G_1}(y)$ (Lemma 22) and then $\{y\} \cup N(y)$ forms a star-cutset. This completes the proof of the claim. $\square$

Case 2: $q = 2$. If there exists an index i $(1 \leqslant i \leqslant p)$ such that A_i contains a vertex x complete for $G_1 \cup G_2$ then $\{x\} \cup G_1 \cup G_2 \cup V(B)$ forms a star-cutset, except if $A_i = \{x\}$ and $p = 2$ but this case cannot occur (Claim 25). Thus, we can assume that, for each i $(1 \leqslant i \leqslant p)$, $A_i = A_i^1 \cup A_i^2$ (where $A_i^1 \subseteq A_i$ is the subset of vertices in A_i which are incomplete for G_1, $A_i^2 = A_i \setminus A_i^1$).

Claim 26. *$\forall i$ $(1 \leqslant i \leqslant p)$ $A_i^1 = \emptyset$ or $A_i^2 = \emptyset$.*

Let i be an index such that $A_i^1 \neq \emptyset$ and $A_i^2 \neq \emptyset$. Let $a \in A_i^1$ and $b \in A_i^2$ such that $ab \in E$ (they exist because A_i is a connected component of A). Let $s \in G_1 \setminus N_{G_1}(a)$ and $r \in G_2 \setminus N_{G_2}(b)$. Then for every u in A_j $(j \neq i)$ the subset $\{r, a, b, s, u\}$ induces a P_5 or a C_5. $\square$

Claim 27. $p = 2$.

Assume that $p \geqslant 3$, so there exist two indices i and j $(i \neq j)$ such that $A_i^1 \neq \emptyset$ and $A_j^1 \neq \emptyset$ (or $A_i^2 \neq \emptyset$ and $A_j^2 \neq \emptyset$). Let $x \in A_i^1 \cup A_j^1$ such that $N(x) \cap G_1$ is maximal (say

$x \in A_i^1$), then

$$\forall y \in A_j^1,\ N_{G_1}(y) \subseteq N_{G_1}(x) \quad \text{(Lemma 22 with } v \in G_2\text{)}.$$

But in this case, $\{x\} \cup N_{G_1}(x) \cup G_2 \cup V(B)$ forms a star-cutset. So,

$$p = 2 \text{ and } A_1 = A_1^1, \quad A_2 = A_2^2. \qquad \square$$

Claim 28. *A_1 and A_2 induce complete subgraphs.*

Assume that A_1 (or A_2) does not induce a complete subgraph. Let $x \in A_1$ be a vertex such that x is not adjacent to all the other vertices in A_1 and such that $N(x) \cap G_1$ is maximal for inclusion. Let $T \subset A_1$ be the subset of vertices in A_1 which are not adjacent to x, then

$$\forall t \in T, \quad N_{G_1}(t) \subseteq N_{G_1}(x) \quad \text{(Lemma 22 with } v \in G_2\text{)}.$$

But $\{x\} \cup N_{G_1}(x) \cup G_2 \cup N_{A_1}(x) \cup V(B)$ forms a star-cutset. This completes the proof of the claim. $\square$

Now we assume that $|A_1| \geqslant |A_2|$.

(1) $|A_2| \geqslant 2$. Let $v, w \in A_2$ and let K be an ω-clique of G containing v but not w (Proposition 2).
  - If $K \subseteq G_1 \cup A_2 \cup V(B)$ then one can find a clique of size $\omega + 1$ by adding vertex w to K (w is adjacent to all vertices from G_1, A_2 and $V(B)$).
  - If $K \subseteq G_2 \cup A_2 \cup V(B)$, let K' be the clique formed by $K \cap G_2$, $K \cap V(B)$ and A_1 (all the vertices in A_1 are adjacent to all vertices from G_2 and $V(B)$), then K' is a clique of size strictly greater than ω (since $|A_1| > |A_2| - 1 \geqslant |K \cap A_2|$).

(2) $|A_2| = 1$. We know that A_1 is a clique and that all vertices in A_1 are adjacent to all vertices from G_2 and $V(B)$, so, if $|A_1| \geqslant 2$ we have $\omega([G_2]_G) \leqslant \omega - 2$. Then let $v \in G_2$ and $x \in A_1$, we cannot find an ω-clique that contains v and not x which contradicts Proposition 2.

So $|A_1| = |A_2| = 1$ (say $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$). But in this case $\{a_1, a_2\}$ forms an even-pair which contradicts Lemma 3. $\square$

Theorem 29 (Seinsche [14]). *A graph G is P_4-free if and only if for all induced subgraph H of G with more than one vertex, H or $\bar{H}$ is disconnected.*

Proof of Theorem 13. Assume that $[S]_G$ is P_4-free, since $|S| \geqslant 2$ and $[S]_G$ is connected (Theorem 12), we know that $[S]_{\bar{G}}$ is disconnected (Theorem 29). There exists a partition of S in two nonempty subsets A and B such that

$$\forall a \in A, \quad \forall b \in B, \qquad ab \in E.$$

Let $S = A \cup B$ be such a partition (with A taken as small as possible).

- If $|A| = 1$, then S forms a star-cutset.

- If $|A| \geqslant 2$, then the subgraph induced by subset A is disconnected (else A is not minimal) which contradicts Lemma 10. $\square$

Proof of Theorem 19. Since G is a minimal imperfect and (P_5, C_5)-free graph, we know that $[N(v)]_G$ is connected (Corollary 14). Let w be a vertex of $N(v)$ such that $\mathscr{M}(w)$ is maximal for inclusion. Let $Y = N(v) \setminus (\{w\} \cup N_{N(v)}(w))$. We have $Y \neq \emptyset$, else any ω-clique containing v also contains w, which is impossible. Now, we show that $M = \mathscr{M}(w)$. We know that $\forall y \in Y\ \mathscr{M}(y) \subseteq \mathscr{M}(w)$ (Lemma 22), so if $M \setminus \mathscr{M}(w) \neq \emptyset$, the set $\{w\} \cup N_{N(v)}(w) \cup \mathscr{M}(w)$ is a star disconnecting subgraphs induced by subsets $M \setminus \mathscr{M}(w)$ and Y. We also have $[Y]_G$ is connected else $\{v\} \cup \{w\} \cup N_{N(v)}(w) \cup \mathscr{M}(w)$ is a star-cutset. $\square$

Before proving Lemma 21, we recall some properties of bipartite connected graphs without induced P_5.

Let A be a connected bipartite graph on $V(A) = V_1 \cup V_2$. Let $x \in V_1$, for short we put $N_2(x) = N_{V_2}(x)$. The following three properties are equivalent:

(1) $\forall x, x' \in V_1,\ N_2(x) \subseteq N_2(x')$ or $N_2(x') \subseteq N_2(x)$.
(2) A contains no induced P_5.
(3) $\forall y, y' \in V_2,\ N_1(y) \subseteq N_1(y')$ or $N_1(y') \subseteq N_1(y)$.

Remark 30. Property (1) shows that neighbourhoods, in V_2, of vertices in V_1 are pairwise comparable which allow us to order them such that

$$N_2(x_i) \subseteq N_2(x_j) \quad \text{iff } i \geqslant j;$$

we do the same for vertices in V_2. One can remark that $N_2(x_1) = V_2$ (A is connected).

We denote by $\mathscr{U}_1$ (resp. $\mathscr{U}_2$) the subset of vertices in V_1 (resp. V_2) which are universal for V_2 (resp. V_1) and $\overline{\mathscr{U}_1}$ (resp. $\overline{\mathscr{U}_2}$) the other vertices in V_1 (resp. V_2).

As before, let G be a graph and let S be a minimal cutset of G. We denote by $G_1, G_2, \ldots, G_q$ ($q \geqslant 2$) the connected components of $G \setminus S$ and we say that a vertex in S is

- of type U, if it is complete for $G_1, G_2, \ldots, G_q$;
- of type G_i, if it is incomplete for G_i ($1 \leqslant i \leqslant q$).

Lemma 31. *Let G be a minimal imperfect and (P_5, C_5)-free graph and let S be a minimal cutset of G. If xy is an edge of $[S]_G$ such that x is of type G_i and y of type G_j $(i \neq j)$ then for every vertex $z \in S$ we have $xz \in E$ or $yz \in E$.*

Proof. Assume there exists a vertex z in S such that $xz \notin E$ and $yz \notin E$. Let $u \in G_i \backslash N_{G_i}(x)$ and $v \in G_j \backslash N_{G_j}(y)$ then the subset $\{u, y, x, v, z\}$ induces a P_5 or a C_5. $\square$

Proof of Lemma 21. We assume that A (with $V(A)=V_1\cup V_2$) is a bipartite graph and that S is the complete join of A and $B=S\setminus A$. We know that A is connected (Lemma 10) and we claim that A is not a complete bipartite graph.

Indeed, if A is a complete bipartite graph (i.e. A is the complete join of two graphs A_1 and A_2 without edges, where $V(A)=V(A_1)\cup V(A_2)$) then:

- If $|V(A_1)|=1$, then S forms a star-cutset.
- If $|V(A_1)|\geqslant 2$, then S is the complete join of the two graphs A_1 and $A_2\cup B$ (where $G_1\cup G_2=(V(G_1)\cup V(G_2),E(G_1)\cup E(G_2))$, for two graphs G_1 and G_2) but $V(A_1)$ is a stable set which contradicts Lemma 10.

We order vertices in A like in Remark 30. We know that $\mathscr{U}_1\neq\emptyset$ (resp. $\mathscr{U}_2\neq\emptyset$) and since the bipartite graph is not complete we have $\overline{\mathscr{U}_1}\neq\emptyset$ (resp. $\overline{\mathscr{U}_2}\neq\emptyset$).

Claim 32. • $|\mathscr{U}_1|\leqslant 2$ *and* $|\mathscr{U}_2|\leqslant 2$ *if* $q=2$.
• $|\mathscr{U}_1|=|\mathscr{U}_2|=1$ *if* $q\geqslant 3$.

The vertices in $\mathscr{U}_1$ cannot be of type U, otherwise they would have comparable neighbourhoods with vertices in $\overline{\mathscr{U}_1}\neq\emptyset$ (Lemma 5).

If $q\geqslant 3$ and $|\mathscr{U}_1|\geqslant 2$, let $x_1\in\mathscr{U}_1$ (resp. $x_2\in\mathscr{U}_1$) be of type G_i (resp. G_j). Since $q\geqslant 3$, there exists $k\neq i,j$ such that x_1 and x_2 are complete for G_k. Thus Lemma 22 with $v\in G_k$ implies that $i=j$, but in this case x_1 and x_2 would have comparable neighbourhoods.

If $q=2$, two distincts vertices (in $\mathscr{U}_1$) would be of different types (i.e. incomplete for different components), else they would be comparable. So, in this case, $|\mathscr{U}_1|\leqslant 2$. □

Claim 33. *If* $x\in\mathscr{U}_1$ (*resp.* $\mathscr{U}_2$) *is complete for* G_i, *then every vertex* v *in* $\overline{\mathscr{U}_2}$ (*resp.* $\overline{\mathscr{U}_1}$) *is complete for* G_i.

Otherwise, assume that $v\in\overline{\mathscr{U}_2}$ is not complete for G_i, then there exists $x'\in\overline{\mathscr{U}_1}$ such that $x'v\notin E$ and $x'x\notin E$ which contradicts Lemma 31. □

Claim 34. $|\overline{\mathscr{U}_1}|\leqslant 2$ *and* $|\overline{\mathscr{U}_2}|\leqslant 2$

Let $x\in\mathscr{U}_1$ be complete for G_i, then there exist at most two vertices in $\overline{\mathscr{U}_2}$ which are complete for G_i. Indeed, assume that $\{y_1,y_2,y_3\}\subseteq\overline{\mathscr{U}_2}$ are complete for G_i. We know that the clique number of the graph induced by $G_i\cup S$ is ω (any vertex of G_i belongs to ω ω-cliques); moreover, the clique number of the graph induced by $G_i\cup V(B)$ is $\omega-2$ (since $xy_i\in E(A)$ $(i=1,2,3)$ and both x and y_i $(i=1,2,3)$ are adjacent to all vertices from G_i and B). But this implies that y_1,y_2,y_3 are pairwise co-critical which contradicts Lemma 8 (G is Berge). So, $|\overline{\mathscr{U}_2}|\leqslant 2$ because Claim 33 implies that every vertex in $\overline{\mathscr{U}_2}$ is complete for G_i. □

Claim 35. *If $|\mathscr{U}_1| = 2$ (resp. $|\mathscr{U}_2| = 2$) then $|\overline{\mathscr{U}_2}| = 1$ (resp. $|\overline{\mathscr{U}_1}| = 1$) and this vertex is of type U.*

Indeed, since $|\mathscr{U}_1| = 2$ we have $q = 2$ (Claim 32) and $\mathscr{U}_1$ contains a vertex complete for G_1 and another complete for G_2. This implies that all vertices in $\overline{\mathscr{U}_2}$ are of type U (Claim 33). But if $|\overline{\mathscr{U}_2}| = 2$, those two vertices have comparable neighbourhoods. This completes the proof of the claim. □

Now, we can finish the proof. First, we assume that $\overline{\mathscr{U}_2}$ contains a vertex of type U (say y). A maximal stable set containing y will be included in the bipartite subgraph induced by $V(A)$ (because y is complete for $G_1, G_2, \ldots, G_q$ and B). So this stable set will be of size at most $|\overline{\mathscr{U}_1}| + |\overline{\mathscr{U}_2}|$ or $|\mathscr{U}_2| + |\overline{\mathscr{U}_2}|$ (i.e. of size at most 4). If this stable set is of size 3 or less, we can conclude using Tucker's Theorem [15]. So, it remains two cases to study:

Case 3 $|\overline{\mathscr{U}_1}| = |\overline{\mathscr{U}_2}| = 2$: We can suppose that there exist no edges (in G) between vertices in $\overline{\mathscr{U}_1}$ and vertices in $\overline{\mathscr{U}_2}$ (else the stable set could not be of size 4). But the two vertices in $\overline{\mathscr{U}_2}$ have comparable neighbourhoods, a contradiction.

Case 4 $|\mathscr{U}_2| = |\overline{\mathscr{U}_2}| = 2$: There exist no edges between the vertex in $\overline{\mathscr{U}_1}$ (Claim 35) and those in $\overline{\mathscr{U}_2}$ (because vertices in $\overline{\mathscr{U}_2}$ are not adjacent to all $\mathscr{U}_1 \cup \overline{\mathscr{U}_1}$). In this case, the two vertices in $\overline{\mathscr{U}_2}$ have comparable neighbourhoods, a contradiction.

So neither $\overline{\mathscr{U}_1}$ nor $\overline{\mathscr{U}_2}$ contain vertices of type U. This implies that (Claim 35)

$$|\mathscr{U}_1| = |\mathscr{U}_2| = 1.$$

We note $\mathscr{U}_1 = \{x_1\}$, $\mathscr{U}_2 = \{y_1\}$, $\overline{\mathscr{U}_1} = \{x_2\}$ or $\{x_2, x_3\}$, $\overline{\mathscr{U}_2} = \{y_2\}$ or $\{y_2, y_3\}$, $V_1 = \mathscr{U}_1 \cup \overline{\mathscr{U}_1}$ and $V_2 = \mathscr{U}_2 \cup \overline{\mathscr{U}_2}$.

Suppose that x_1 is of type G_i, then the vertices in $\overline{\mathscr{U}_2}$ are of type G_i (Claim 33 and there exist no vertices of type U). If y_1 is of type G_j ($j \neq i$) then the vertices in $\overline{\mathscr{U}_1}$ are of type G_j, moreover, if $|\overline{\mathscr{U}_1}| = 2$ we have $N_{G_j}(x_2) \subseteq N_{G_j}(x_3)$ (or $N_{G_j}(x_3) \subseteq N_{G_j}(x_2)$) and $N_{G_j}(x_{|V_1|}) \neq G_j$ (because there is no vertices of type U). But, then $\{x_1\} \cup V_2 \cup V(B) \cup N_{G_j}(x_{|V_1|}) \cup \bigcup_{k \neq i,j} G_k$ forms a star disconnecting subgraphs induced by the sets $G_j \setminus N_{G_j}(x_{|V_1|})$ and $\overline{\mathscr{U}_1}$. So, y_1 and the vertices in $\overline{\mathscr{U}_1}$ are of type G_i and all the vertices x_k ($1 \leqslant k \leqslant |V_1|$) are complete for G_j ($j \neq i$). Thus Lemma 22 with $v \in G_j$ and the fact that no two vertices can have comparable neighbourhoods imply: $N_{G_i}(x_k) \subset N_{G_i}(x_l)$ for $k < l$.

The same holds for the vertices y_k. We have $x_{|V_1|} y_{|V_2|} \notin E$ and, $x_{|V_1|}$ and $y_{|V_2|}$ are of type G_i. So Lemma 22 (with $v \in G_k$, $k \neq i$) implies $N_{G_i}(x_{|V_1|}) \subseteq N_{G_i}(y_{|V_2|})$ or $N_{G_i}(y_{|V_2|}) \subseteq N_{G_i}(x_{|V_1|})$, say $N_{G_i}(y_{|V_2|}) \subseteq N_{G_i}(x_{|V_1|})$. Since the vertex $x_{|V_1|}$ is not complete for $G_1, \ldots, G_q$, the subset $\{x_{|V_1|}\} \cup N_{G_i}(x_{|V_1|}) \cup \bigcup_{j \neq i} G_j \cup V(B)$ is a star-cutset of G. This completes the proof of the Lemma. □

Theorem 36. *Let G be a P_5-free minimal imperfect Berge graph and let S be a minimal cutset of G. Then S cannot induce a graph in $\mathscr{B}^*$.*

Proof. Assume that $[S]_G \in \mathscr{B}^*$. We know that $[S]_G$ is connected, then $[S]_G$ is either a bipartite graph (which contradicts Lemma 21 with $A = [S]_G$ and $V(B) = \emptyset$) or the complete join of two graphs G_1 and G_2. In the latter case, assume that $[S]_G$ is the complete join of G_1 and G_2 such that G_1 was not decomposable by the complete join operation. Then, either G_1 is a bipartite graph (contradicts Lemma 21) or G_1 is a disconnected subgraph of $[S]_G$ (contradicts Lemma 10). □

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